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PERTURBATION METHOD TO ANALYZE THE ELASTODYNAMIC FIELD NEAR A KINKED CRACK

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Abstract—A perturbation procedure is presented which expresses the elastodynamic fields near a kinked crack in terms of a power series of κ, where $\kappa\pi$ is the kinking angle. Mode-III and Mixed Mode I–II crack kinking of a semi-infinite crack have been considered. Each order in the approximation requires the field for a crack which propagates in its own plane, but where the crack faces are subjected to crack-face tractions which are related to the actual crack-kinking geometry. For crack kinking under stress-wave loading it was shown in an earlier paper that the lowest order approximation gives very good agreement with exact analytical and numerical results over a substantial range of kinking angles.

1. INTRODUCTION

In [1], an approximate method was used to compute elastodynamic stress intensity factors for Mode-III and Mixed-Mode I–II crack kinking, for cases that the particle velocities are self-similar. It was shown that for an important range of kinking angles the elastodynamic stress intensity factor for a kinked crack can be approximated by the stress intensity factor for the crack propagating in its own plane, provided that the new crack faces are subjected to appropriate surface tractions. For Mode-III cases the approximation was compared with exact results which can be found in an article by Dempsey *et al.* [2]. For Mixed-Mode I – II cases, comparisons with the numerical results of [3] were carried out, and satisfactory agreement was obtained.

In this study, we investigate the mathematical foundation of the approximation of [1]. It is shown that the approximate results correspond to the zeroth order terms in a perturbation procedure for small kinking angle $\kappa\pi$, regardless of whether the problems have self-similar field variables.

For Mode-III the approximate results of [1] showed surprisingly good agreement up to large values of the kinking angle. The reason for this good agreement is revealed by the results of the present study. For Mode-III cases, displacement solutions to the first order system of equations are symmetric with respect to the crack plane, and hence first order contributions to Mode-III stress intensity factors vanish. Consequently, in the approach of [1], which corresponds to the zeroth order approximation, the Mode-III elastodynamic stress intensity factors are accurate up to $0(\kappa^2)$. The first order Mixed-Mode I–II elastodynamic stress intensity factors do, however, not vanish, and hence the results of [1] are accurate only up to $0(\kappa)$.

In the formulation of this study the crack faces are subjected to surface tractions. These can be chosen in such a manner that superpositions yield solutions for traction-free crack faces.

2. MODE-III KINKED CRACK

In a homogeneous, isotropic and linearly elastic solid the antiplane displacement $w(r, \theta, t)$ is governed by

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = s_T^2 \ddot{w}. \tag{2.1}$$

Here r and θ are polar coordinates centered at the original crack tip, $(\dot{\ })= \partial/\partial t$ and s_T is the slowness of transverse waves

$$s_T = 1/c_T, \qquad c_T^2 = (\mu/\rho). \tag{2.2}$$

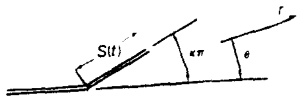

Fig. 1. Kinked crack geometry.

For $t \geq 0$ we consider the following conditions on the crack faces

$$\theta = \pm\pi,\ r > 0: \qquad (\mu/r)\partial w/\partial\theta = \tau_1(r, t) \tag{2.3a}$$

$$\theta = \kappa\pi \pm 0,\ 0 < r < S(t): \qquad (\mu/r)\partial w/\partial\theta = \tau_2(r, t). \tag{2.3b}$$

In addition we have the initial conditions

$$w(r, \theta, t) = \dot{w}(r, \theta, t) = 0 \qquad \text{for } t < 0. \tag{2.4}$$

We will be particularly interested in the elastodynamic Mode-III stress intensity factor for the kinked crack, which is defined as

$$K_{\text{III}}(\kappa, t) = \lim_{(r-S)\to 0^+} [2\pi(r - S)]^{1/2}\sigma_{\theta z}(r, \theta, t)\,|_{\theta=\kappa\pi}. \tag{2.5}$$

The geometry is shown in Fig. 1.

For arbitrary $c_F \equiv \mathrm{d}S/\mathrm{d}t > 0$, an exact solution to (2.1)–(2.4) is not available. In this study an approximate solution is constructed on the basis of a mapping from θ to the new variable χ by:

$$\chi = \frac{\kappa}{\pi(1 - \kappa^2)}\theta^2 + \theta - \frac{\kappa\pi}{1 - \kappa^2} \tag{2.6}$$

or

$$\theta = -\frac{\kappa}{\pi}\chi^2 + \chi + \kappa\pi. \tag{2.7}$$

Equations (2.6) and (2.7) imply that $\theta = -\pi$, $\kappa\pi$ and π are mapped into $\chi = -\pi$, 0, and π, respectively. The displacement $w(r, \chi, t)$ is then found to satisfy the equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \chi^2} - s_T^2\ddot{w} = \frac{-2\kappa}{\pi(1 - \kappa^2)r^2}\left\{\frac{\partial w}{\partial \chi} + 2\left(\chi + \frac{\kappa\pi}{1 - \kappa^2}\right)\frac{\partial^2 w}{\partial \chi^2}\right\} \tag{2.8}$$

with the boundary conditions

$$\chi = \pm\pi,\ 0 < r: \qquad \frac{\mu}{r}\left[\frac{\partial w}{\partial \chi} \pm \frac{2\kappa}{1 - \kappa^2}\frac{\partial w}{\partial \chi}\right] = \tau_1(r, t) \tag{2.9}$$

$$\chi = \pm 0,\ 0 < r < S(t): \qquad \frac{\mu}{r}\left[\frac{\partial w}{\partial \chi} + \frac{2\kappa^2}{1 - \kappa^2}\frac{\partial w}{\partial \chi}\right] = \tau_2(r, t). \tag{2.10}$$

Let us now consider a perturbation series for $w(r, \chi, t)$ of the general form

$$w(r, \chi, t) = w^{(0)}(r, \chi, t) + \kappa\pi w^{(1)}(r, \chi, t) + \cdots \tag{2.11}$$

By using the definition of $K_{\text{III}}(\kappa, t)$ given by eqn (2.5) it is apparent that the expansion corresponding to (2.11) is

$$K_{\text{III}}(\kappa, t) = K_{\text{III}}^{(0)} + \kappa\pi K_{\text{III}}^{(1)} + \cdots \tag{2.12}$$

By substituting (2.11) into (2.8)–(2.10) and collecting terms of the zeroth power of $\kappa\pi$, we obtain for $w^{(0)}(r, \chi, t)$

$$\frac{\partial^2 w^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial w^{(0)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w^{(0)}}{\partial \chi^2} = s_T^2 \ddot{w}^{(0)} \tag{2.13}$$

$$\chi = \pm\pi,\ 0 < r: \qquad (\mu/r)\partial w^{(0)}/\partial\chi = \tau_1(r, t) \tag{2.14}$$

$$\chi = \pm 0,\ 0 < r < S(t): \qquad (\mu/r)\partial w^{(0)}/\partial\chi = \tau_2(r, t). \tag{2.15}$$

The Mode-III elastodynamic fields for the problem defined by (2.13)–(2.15), subject to zero initial conditions, can be obtained by the method of [4]. In terms of the crack-face tractions, the stress field in the plane of the crack, ahead of the crack tip, can be expressed as:

$$\frac{\mu}{r}\frac{\partial w^{(0)}}{\partial\chi}(r, \chi = 0, t; S(t_1)) = \frac{1}{\pi}\frac{1}{[r - S(t_1)]^{1/2}}\int_{r-c_Tt}^{S(t_1)} \frac{f^{(0)}(v, t - s_Tr + s_Tv)[S(t_1) - v]^{1/2}}{r - v}\,\mathrm{d}v \tag{2.16}$$

where t_1 must be computed from

$$c_Tt - r = c_Tt_1 - S(t_1), \tag{2.17}$$

and

$$f^{(0)}(v, \tau) = \tau_1(v, \tau)H(-v) - \tau_2(v, \tau)H(v)H[S(\tau) - v]. \tag{2.18}$$

The displacement field can be expressed in terms of both the crack face tractions and the stress field ahead of the crack tip by

$$w^{(0)}(r, \chi, t; S(t_1)) = \frac{c_T}{\pi\mu}\iint_R \frac{q(\bar{x}, \bar{t})\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{t}}{[c_T^2(t - \bar{t})^2 - (r\cos\chi - \bar{x})^2 - r^2\sin^2\chi]^{1/2}}, \tag{2.19}$$

where R is the domain of dependence in the time-space domain, defined by

$$c_T(t - \bar{t}) - [(r\cos\chi - \bar{x})^2 + r^2\sin^2\chi]^{1/2} \geq 0, \qquad t \geq \bar{t} \geq 0, \tag{2.20a,b}$$

and

$$q(\bar{x}, \bar{t}) = f^{(0)}(\bar{x}, \bar{t}) + \frac{\mu}{r}\frac{\partial w^{(0)}}{\partial\chi}(\bar{x}, \chi = 0, \bar{t})H[\bar{x} - S(\bar{t})]. \tag{2.21}$$

The stress intensity factor follows from (2.5) and (2.16) as

$$K_{\mathrm{III}}^{(0)} = \sqrt{\frac{2}{\pi}}(1 - c_F/c_T)^{1/2}\int_{S-c_Tt}^{S}(S - v)^{-1/2}f^{(0)}(v, t - s_TS + s_Tv)\,\mathrm{d}v. \tag{2.22}$$

Equations (2-13)–(2.15) show that in first approximation elastodynamic stress intensity factors for Mode-III crack kinking problems can be computed for a crack propagating straight ahead, provided that the crack face tractions are the ones corresponding to the actual kinked geometry.

By collecting terms of order κ, the equations governing the first order problem in the perturbation procedure are obtained as

$$\frac{\partial^2 w^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial w^{(1)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w^{(1)}}{\partial \chi^2} - s_T^2 \ddot{w}^{(1)} = \frac{-1}{r^2}\frac{2}{\pi^2}\left(\frac{\partial w^{(0)}}{\partial \chi} + 2\chi \frac{\partial^2 w^{(0)}}{\partial \chi^2}\right) \tag{2.23}$$

$$\chi = \pm\pi,\ 0 < r: \qquad (\mu/r)\partial w^{(1)}/\partial\chi = \mp(2/\pi)\tau_1(r, t) \tag{2.24}$$

$$\chi = \pm 0,\ 0 < r < S(t): \qquad (\mu/r)\partial w^{(1)}/\partial\chi = 0. \tag{2.25}$$

Since $w^{(0)}(r, \chi, t)$ is an anti-symmetric field with respect to $\chi = 0$, $\partial w^{(0)}/\partial\chi$ and $\chi\partial^2 w^{(0)}/\partial\chi^2$ are symmetric with respect to $\chi = 0$. It then follows from (2.23)–(2.25) that $w^{(1)}$ is also a symmetric field with respect to $\chi = 0$, hence $K_{III}^{(1)} \equiv 0$.

It is noted from (2.16) and (2.19) that the displacements and the stresses of the zeroth order problem depend explicitly only on the crack-kink length, but not on the crack tip speed. For the higher order problems the stress fields in the plane of the crack, ahead of the crack tip, are of the same form as (2.16), except that $f^{(0)}$ in (2.16) must be replaced by appropriate functions $f^{(n)}$:

$$K_{III}^{(n)} = \sqrt{\frac{2}{\pi}}(1 - c_F/c_T)^{1/2} \int_{S - c_T t}^{S} (S - v)^{-1/2} f^{(n)}(v, t - s_T S + s_T v)\, dv \tag{2.26}$$

where n is even, while $K_{III}^{(n)} \equiv 0$ for n is odd. Thus K_{III} depends on the crack tip speed only through the term $(1 - c_F/c_T)^{1/2}$.

3. MIXED MODE I–II KINKED CRACK

A complete statement of the equations which govern plane-strain elastodynamic fields is given in [5]. In the usual manner the displacement components can be expressed in terms of displacement potentials $\phi(r, \theta, t)$ and $\psi(r, \theta, t)$ by:

$$u_r = \frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{\partial\psi}{\partial r}, \tag{3.1a,b}$$

where ϕ and ψ satisfy uncoupled wave equations

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = s_L^2\ddot{\phi}, \tag{3.2a}$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = s_T^2\ddot{\psi}. \tag{3.2b}$$

Here s_T is defined by (2.2) and s_L is the slowness of longitudinal waves

$$s_L = 1/c_L, \qquad c_L^2 = (\lambda + 2\mu)/\rho. \tag{3.3}$$

For $t \geq 0$ we consider the folowing conditions on the crack faces

$$\theta = \pm\pi,\ r > 0: \qquad \sigma_\theta = p_1(r, t) \tag{3.4a}$$

$$\sigma_{\theta r} = \tau_1(r, t) \tag{3.4b}$$

$$\theta = \kappa\tau \pm 0,\ 0 < r < S(t): \quad \sigma_\theta = p_2(r, t) \tag{3.5a}$$

$$\sigma_{\theta r} = \tau_2(r, t). \tag{3.5b}$$

In addition we have the initial conditions

$$t < 0: \qquad \phi(r, \theta, t) = \dot{\phi}(r, \theta, t) = \psi(r, \theta, t) = \dot{\psi}(r, \theta, t) = 0. \tag{3.6}$$

The elastodynamic Mode I and Mode II stress intensity factors for the kinked crack are defined as

$$K_{\rm I}(\kappa, t) = \lim_{(r-S)\to 0^+} [2\pi(r - S)]^{1/2}\sigma_\theta(r, \theta, t)\,|_{\theta=\kappa\pi} \tag{3.7a}$$

$$K_{\rm II}(\kappa, t) = \lim_{(r-S)\to 0^+} [2\pi(r - S)]^{1/2}\sigma_{\theta r}(r, \theta, t)\,|_{\theta=\kappa\pi}. \tag{3.7b}$$

Let us now consider perturbation series for $\phi(r, \chi, t)$ and $\psi(r, \chi, t)$ of the general form:

$$H(r, \chi, t) = H^{(0)}(r, \chi, t) + \kappa\pi H^{(1)}(r, \chi, t) + \cdots, \qquad H = \phi \text{ or } \psi. \tag{3.8}$$

By using the definition of $K_{\rm I}(\kappa, t)$ and $K_{\rm II}(\kappa, t)$ given by eqn (3.7) it is apparent that the expansions corresponding to (3.8) are

$$K_i(\kappa, t) = K_i^{(0)} + 0(\kappa), \qquad i = \text{I or II}. \tag{3.9}$$

From (2.6)–(2.7), (3.1)–(3.2), (3.4)–(3.6) and (3.8), we obtain for $\phi^{(0)}(r, \chi, t)$ and $\psi^{(0)}(r, \chi, t)$

$$\frac{\partial^2\phi^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial\phi^{(0)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi^{(0)}}{\partial\chi^2} = s_L^2\ddot{\phi}^{(0)} \tag{3.10a}$$

$$\frac{\partial^2\psi^{(0)}}{\partial r^2} + \frac{1}{r}\frac{\partial\psi^{(0)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi^{(0)}}{\partial\chi^2} = s_T^2\ddot{\psi}^{(0)} \tag{3.10b}$$

$$\chi = \pm\pi, r > 0: \qquad \sigma_\chi^{(0)} = p_1(r, t) \tag{3.11a}$$

$$\sigma_{\chi r}^{(0)} = \tau_1(r, t) \tag{3.11b}$$

$$\chi = \pm 0, 0 < r < S(t): \quad \sigma_\chi^{(0)} = p_2(r, t) \tag{3.12a}$$

$$\sigma_{\chi r}^{(0)} = \tau_2(r, t). \tag{3.12b}$$

The Mixed Mode I–II elastodynamic stress intensity factors for the problem defined by (3.10)–(3.12) together with zero initial conditions can be found in [6]. As for the Mode-III problem, the stress fields ahead of the crack tip in the plane of the crack depend explicitly only on the crack-kink length but not the crack tip speed. For the zeroth order case, the stress intensity factors at the kinked crack tip are obtained as

$$K_{\rm I} = (1 - s_Rc_F)(1 - s_Lc_F)^{-1/2}\Gamma_+(-s_F)k_{\rm I}(S, \kappa, t) \tag{3.13}$$

$$K_{\rm II} = (1 - s_Rc_F)(1 - s_Tc_F)^{-1/2}\Gamma_+(-s_F)k_{\rm II}(S, \kappa, t), \tag{3.14}$$

where $s_F = 1/c_F$, s_R is the slowness of Rayleigh surface waves,

$$\Gamma_+(\xi) = \exp\left\{-\frac{1}{\pi}\int_{s_L}^{s_T}\tan^{-1}\left[\frac{4z^2(z^2 - s_L^2)^{1/2}(s_T^2 - z^2)^{1/2}}{(s_T^2 - 2z^2)^2}\right]\frac{dz}{z + \xi}\right\}, \tag{3.15}$$

and

$$k_{\mathrm{I}}(S, \kappa, t) = \sqrt{\frac{2}{\pi}} \int_{S-c_T t}^{S} (S - v)^{-1/2} F_{\chi\chi}(v, t - s_T S + s_T v)\, \mathrm{d}v \tag{3.16a}$$

$$k_{\mathrm{II}}(S, \kappa, t) = \sqrt{\frac{2}{\pi}} \int_{S-c_L t}^{S} (S - v)^{-1/2} F_{\chi r}(v, t - s_L S + s_L v)\, \mathrm{d}v \tag{3.16b}$$

In (3.16):

$$F_{ij}(v, t) = f_{ij}^{(0)}(v, t) + 0(\kappa) \tag{3.17}$$

$$f_{ij}^{(0)}(v, t) = g_{ij}^{*}(v, t) - \frac{\partial}{\partial t}\left\{ \frac{(s_R - s_L)^{1/2}(s_R - s_T)^{1/2}}{\Gamma_{+}(-s_R)} \int_0^{c_R t} \right.$$
$$\times\, g_{ij}^{*}(v - \eta, t - s_R\eta)\, \mathrm{d}\eta - \frac{1}{\pi}\int_{s_L}^{s_T} \frac{(\xi - s_L)^{1/2}(s_T - \xi)^{1/2}}{(\xi - s_R)} \mathrm{Re}\left(\frac{1}{\Gamma_{+}(-\xi)}\right)$$
$$\left. \times \left[\int_0^{t/\xi} \overset{*}{g}_{ij}(v - \eta, t - \xi\eta)\, \mathrm{d}\eta\right] \mathrm{d}\xi \right\}. \tag{3.18}$$

In eqns (3.17)–(3.18), $i, j = r, \chi$, and

$$g_{\chi\chi}^{*}(v, t) = -p_1(v, t)H(-v) - p_2(v, t)H(v)H(S - v) \tag{3.19}$$

$$g_{\chi r}^{*}(v, t) = -\tau_1(v, t)H(-v) - \tau_2(v, t)H(v)H(S - v). \tag{3.20}$$

It can be shown that the contributions to the Mixed-Mode I–II stress intensity factors from the first order problems do generally not vanish. Hence the zeroth order result is valid to $0(\kappa)$. Nevertheless, the results of [1] showed very satisfactory agreement with the numerical results of [3] for values of κ up to $\kappa = 0.25$.

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